2 + 2 = 1

Mike Arnautov, August 2012

The bold title of this essay may well prompt the reader to enquire whether I really mean that if one puts two apples on a table and then adds another two, the outcome would be (or could be) a single apple. The answer is, of course, no – that would be absurd.

The reader may further point out that some crude redefinition of symbols such as '2', '1', '+' and '=' cannot be of any philosophical significance. I agree. I have no intention of arguing my case on any such grounds. I am not redefining '2' to mean one half, or '+' to mean division, or '=' to mean greater than, nor is any other such trivial trickery involved.

With these two potential misunderstandings out of the way, let me state my case in more detail than can be done in an eye-catching title. What I intend to argue is that there is a legitimate mathematical sense in which the abstract identity 2 + 2 = N, where N is a natural number (i.e. an integer greater or equal 0) and all other symbols have their usual meanings, could be considered to be valid for N being 4 or 1, and for no other value of N. Or to put it another way, that if 2 + 2 = 4 is a necessary truth, then so is 2 + 2 = 1. Consequently, I contend that our preference for the form of arithmetic in which 2 + 2 = 4 is motivated by empirical considerations, rather than by the demands of abstract necessity.

It should be noted that I use the expression *necessary truth* in its philosophically common meaning of something which is true in all possible worlds[1]. If the reader prefers any other conception of necessary truth, the discussion below may or may not apply.

The arithmetic of natural numbers clearly has its roots in counting objects. Hence any defence of 2 + 2 = 4 must deal with the obvious counter-examples of worlds without discrete objects (the extreme case of a completely empty world is, arguably, ruled out by Quantum Mechanics) and of worlds in which discrete objects exist (to whatever value of 'exist') but cannot be assigned individual stable identities.

Might one perhaps appeal to measuring instead of counting to deal with the possibility of there being no countable objects?[2] This seems very doubtful. Obvious measures of a geometric nature (length, area, volume...) run into trouble in closed non-Euclidean worlds because for sufficiently large units of such measures their sum can exceed the maximum possible in such a world, causing 'wrap-around' effects. If one looks beyond geometry to more physically based measurements, between them theories of Special Relativity, General Relativity and Quantum Mechanics provide plenty of scope for challenging the notion of simple scale-independent additivity. In any case, we must heed Einstein's main philosophical lesson that there can be no measurement without a clearly specified measuring procedure. How is one to measure in a world with no identifiable objects?

The obvious response to these difficulties is to appeal to the abstract definition of common-sense arithmetic in the axioms of Peano Arithmetic (further just PA). Given the standard set-theoretical derivation of natural numbers starting with nothing more than an empty set[3], it cannot be denied that the abstraction of PA is consistent even with the extreme case of a completely empty world.

I am happy to concede that PA as an axiomatic structure is consistent with any possible (or perhaps even any imaginable) world. The difficulty is that so is any other axiomatically defined mathematical structure. This would not in itself matter, except for the fact that some of such structures share with PA the definition of their basic elements (natural numbers) and of operations

1 The difficult question of what constitutes a 'possible world', as (possibly) opposed to an 'imaginable world' on one hand and 'physically possible world' on the other, is of necessity beyond the limited scope of this essay. Personally I doubt the philosophical usefulness of the notion of 'possible worlds', but I am happy enough to work with it in the limited context of my subject matter.

2 The way our use of numbers got generalised from counting discrete objects to quantifying in general is a fascinating subject in its own right, though I do not propose to examine it here.

3 Zermelo-Fraenkel Set Theory. See, e.g. http://en.wikipedia.org/wiki/Set-theoretic_definition_of_natural_numbers.

performed on those basic elements (addition, multiplication and – implicit in PA – subtraction and division).

The Appendix lists the axioms of a formal definition of PA, and shows how a change to a single axiom transforms it into a definition of modular arithmetic to base N (further just MA(N)), or more popularly, into clock arithmetic, in which numbers 'wrap around' after the first N numbers. For sufficiently large values of N, MA(N)is for all practical purposes indistinguishable from PA.[4] In fact PA can be understood as the limit of MA(N) as N goes to infinity (see the Appendix).

Modular arithmetic to base N is most easily thought of as an equivalent of PA, but with every number and every result of a calculation being replaced by the remainder of integer division of that number by the base number N. This is, in fact, how the MA family was originally defined and studied[5], but this model of modular arithmetic, and indeed the nomenclature of MA(N) arising from it, is in one sense deeply misleading. It suggests that the existence of MA(N) is parasitic on PA, in that we need full PA (or at least an MA to some higher base) to perform our calculations and then work out the result in MA(N). Worse, the very notation of MA(N) uses the number N, which by definition does not exist within the MA(N) arithmetic, since the remainder of integer division of any number by N is necessarily smaller than N. However, as the Appendix shows MA(N) can be defined without any dependency on PA or any MA to a higher base than N.

The key point is that PA axioms are of necessity inductive in their definition of natural numbers and of operators acting on those numbers. This enables PA to capture an infinite structure in a finite set of axioms. Thus natural numbers are defined by defining the lowest natural number (zero in the modern version, even though in the past zero used to be excluded from the natural numbers set), and then demanding that each natural number has a successor (or 'next') natural number – there can be no reference to 'incrementing by 1', because at this stage neither the operation of addition, nor the number one have been defined as yet. The difference between PA and MA(N) is trivially simple. One of the axioms of PA states that there is no natural number such that its successor is 0. In MA(N) this is replaced by the axiom stating that there is a specific natural number (N-1 in PA nomenclature), the successor of which is 0. No other axiom changes are required for the shift from PA to MA(N).

It follows, that MA(N) only has N elements in its set of natural numbers, but those elements are defined in *exactly* the same manner as the corresponding elements of PA. Furthermore, the inductive definitions of addition and multiplication are in no way affected by the above axiom change. Similarly, there is no change to the equality relationship.

Modular arithmetic has been extensively studied by mathematicians, and it is well known that not all MA(N) structures are 'well behaved'. The axioms of PA do not define inverse operations of subtraction and division, for the simple reason that inversion of addition and multiplication is unproblematic in PA. By contrast, in MA(N) division need not be well defined. E.g. in MA(12) 4 * 1 = 4, but so is 4 * 4, hence 4 / 4 could be either 1 or 4. However, such problems only arise when N is not a prime number and thus can be dealt with by considering only modular arithmetics to a prime base, of which there are still infinitely many.

We are now ready to tackle the equation 2 + 2 = N in modular arithmetic. On the one hand, in order for MA(N) to have number 2 at all, N must be greater than 2. On the other hand, if N is greater than 4, then 2 + 2 = 4. For any anomalies to arise, we must have 2 < N <= 4, where N is a prime number. And indeed, 3 is a prime number satisfying the condition and 2 + 2 = 1 in MA(3).

Thus we have:

- MA(N) (where N is a prime) is a well defined arithmetic, sharing *all* of the PA axioms except the one which guarantees there being an infinite number of natural numbers.

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4 For N exceeding by many orders of magnitude the estimated number of atoms in the visible universe, it is unclear how one could distinguish between PA and MA(N) even in principle. In practice, one does not need to go to such extremes. Digital computers embody a variant of modular arithmetic for much lover values of N (2^{64} in the case of 64-bit architectures – this may not be a prime number, but division trouble is avoided by adding additional constraints).

5 By Leonard Euler, in about 1750.

- 2 + 2 has no meaning in MA(2), has the value of 1 in MA(3) and the value of 4 for any MA(N) where N is a prime greater than 3 (and also in PA, which is a limit case of MA(N)).
- Since MA(3) is an abstract axiomatic structure, its philosophical status is no different from that of PA, which is also an abstract axiomatic structure: if PA is necessarily true then so is MA(3) (and if not, not).
- 2 + 2 = 4 is necessarily true within the context of PA and necessarily false within the context of MA(3).
- 2 + 2 = 1 is necessarily true within the context of MA(3) and necessarily false within the context of PA.
- Hence if 2 + 2 = 4 is deemed to be necessarily true, then there is no *a priori* reason to deny the same status to 2 + 2 = 1.
- However, if something is necessarily true in one context and necessarily false in another, philosophically equivalent context, then it cannot be necessarily true independently of its context..
- It follows that neither 2 + 2 = 4 as such, nor 2 + 2 = 1 as such can be necessarily true.

Thus the reason we prefer PA over MA(N) for relatively small values of N is clearly a pragmatic one: two apples and two apples make four apples, not one. We have no *a priori* reasons for that preference. In other words: 2 + 2 = 4 is an empirical truth.

Appendix

Here are the PA axioms (adapted from Wikipedia[6]):

0. 0 is a natural number.
1. For every natural number x, $x = x$. That is, equality is reflexive.
2. For all natural numbers x and y, if $x = y$, then $y = x$. That is, equality is symmetric.
3. For all natural numbers x, y and z, if $x = y$ and $y = z$, then $x = z$. That is, equality is transitive.
4. For all a and b, if a is a natural number and $a = b$, then b is also a natural number. That is, natural numbers are closed under equality.
5. For every natural number n, $S(n)$ is a natural number. That is, every natural number has an immediately succeeding natural number.
6. For every natural number n, $S(n) = 0$ is false. That is, there is no natural number whose successor is 0.
7. For all natural numbers m and n, if $S(m) = S(n)$, then $m = n$. That is, S is an injection.
8. If K is a set of natural numbers such that 0 is in K and for every natural number, if n is in K, then $S(n)$ is in K, then K contains all natural numbers. That is, mathematical induction works.
9. For every natural number a, $a + 0 = a$. That is, 0 is the null element of addition.
10. For all natural numbers a and b, $a + S(b) = S(a + b)$. That is, addition is defined inductively.
11. For every natural number a, $a * 0 = 0$. That is, 0 is the dominant element of multiplication.
12. For all natural numbers a and b, $a * S(b) = a + (a * b)$. That is, multiplication is defined inductively.

Operators of subtraction and division are straightforward inverses of addition and multiplication respectively, and as such are not explicitly defined by PA axioms. This is possible because in PA the result of dividing a by b is either undefined (there is no c such that $b * c = a$), or uniquely defined (there is just one such c).

The above definition can be modified to define modular arithmetic to base N + 1 by changing Axiom 6 to

6. There is a natural number N such that $S(N) = 0$.

6 http://en.wikipedia.org/wiki/Peano_axioms

In PA, Axioms 6 and 7 between them guarantee that there are infinitely many natural numbers. However, even though the modified Axiom 6 has the effect of there being only N different natural numbers, it does not clash with Axiom 7, because 0 is not defined as a successor of some other natural number.

Note that in the limit of N going to infinity, this substitute Axiom 6 becomes equivalent to the original one, demanding that there is no N for which S(N) = 0. Hence my assertion that PA can be viewed as the limit of MA(N) as N goes to infinity.